An Interval-Valued Axiomatic Utility Theory for Dempster-Shafer Belief Functions

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Outline

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Introduction

- Main goal is to propose an axiomatic utility theory for D-S belief function lotteries similar to vN-M's axiomatic framework for probabilistic lotteries.
- D-S theory consists of representations (basic probability assignments, belief, plausibility, commonality, credal sets) + Dempster's combination rule + marginalization rule.
- Representations are also used in other theories, e.g., in the imprecise probability community, credal sets are used with Fagin-Halpern combination rule.
- Our axiomatic utility theory is designed for the D-S theory.
- Therefore, Dempster's combination must be an integral part of our theory.

Outline

vN-M's Utility Theory

- Let $\mathbf{O} = (O_1, \ldots, O_r)$ denote a finite set of outcomes.
- Let $\mathbf{p} = (p_1, \ldots, p_r)$ denote a probability mass function (PMF) on $\mathbf{O}$, i.e., $p_i \geq 0$ for $i = 1, \ldots, r$, and $\sum_{i=1}^{r} p_i = 1$.
- We call $L = [\mathbf{O}, \mathbf{p}]$ a probabilistic lottery on $\mathbf{O}$. We assume that L will result in one outcome O_i (with prob. p_i), and it is not repeated.
- We are concerned with a decision maker (DM) who has preferences on $\mathcal{L}$, the set of all lotteries on $\mathbf{O}$.
- We write $L \succ L'$ if the DM prefers L to L', $L \sim L'$ if the DM is indifferent between L and L', and $L \succsim L'$ is the DM either prefers L to L' or is indifferent between the two.
- Our task is to find a real-valued utility function $u : \mathcal{L} \to \mathbb{R}$ such that if $L \succ L'$, then $u(L) > u(L')$, and if $L \sim L'$, then $u(L) = u(L')$.

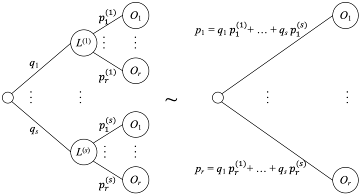
vN-M's Utility Theory

- Of course, this is not always possible (e.g., Condorcet paradox). But, if the DM's preferences satisfy some assumptions, then we can construct such a utility function.
- A utility function is said to be linear if $u([\mathbf{O}, \mathbf{p}]) = \sum_{i=1}^{r} p_i \, u(O_i)$, where O_i can be considered as a degenerate lottery where $p_i = 1$.
- von Neumann-Morgenstern's (vN-M's) utility theory was first published in 1947 in an appendix of the 2^{nd} edition of *Theory of Games & Economic Behavior*.
- There are several axiomatizations of vN-M's utility theory by Herstein-Milnor [1953], Hausner [1954], Luce-Raiffa [1957], Jensen [1967], Fishburn [1982], etc. We will describe the one by Luce-Raiffa [1957].

vN-M's Utility Theory

- **Assumption** $1p$ (ordering of outcomes). For any two outcomes O_i and O_j, either $O_i \succsim O_j$ or $O_j \succsim O_i$. Also, if $O_i \succsim O_j$ and $O_j \succsim O_k$, then $O_i \succsim O_k$. Thus, ordering $\succsim$ over **O** is complete and transitive.
- Given Assumption $1p$, we can label the outcomes so that $O_1 \succsim O_2 \succsim \ldots \succsim O_r$.
- To avoid trivialities, we assume $O_1 \succ O_r$.

vN-M's Utility Theory



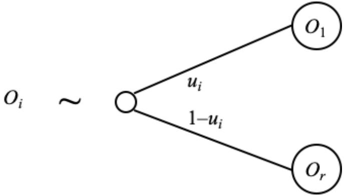
- **Assumption** $2p$ (reduction of compound lotteries). Any compound lottery $[\mathbf{L}, \mathbf{q}]$ (where $\mathbf{L} = (L^{(1)}, \ldots, L^{(s)})$, and $L^{(i)} = [\mathbf{O}, \mathbf{p}^{(i)}]$) is indifferent to a simple (non-compound) lottery $[\mathbf{O}, \mathbf{p}]$, where

$$p_i = q_1\, p_i^{(1)} + \ldots + q_s\, p_i^{(s)} \tag{1}$$

- PMF $\mathbf{p}^{(i)}$ is a conditional PMF for $\mathbf{O}$ given that lottery $L^{(i)}$ is realized in the first stage.
- The PMF $\mathbf{p} = (P(\mathbf{L}) \otimes P(\mathbf{O}|\mathbf{L}))^{\downarrow \mathbf{O}}$ is the marginal of the joint PMF for $\mathbf{O}$.

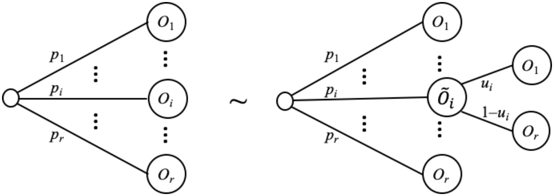
vN-M's Utility Theory

- A lottery $[(O_1, O_r), (u, 1-u)]$ with only two outcomes O_1 and O_r, with PMF $(u, 1-u)$ is called a reference lottery. Let $\mathbf{O}_2$ denote (O_1, O_r).
- **Assumption** $3p$ (continuity) Each outcome O_i is indifferent to a reference lottery $[\mathbf{O}_2, (u_i, 1-u_i)]$ for some $0 \leq u_i \leq 1$, i.e., $O_i \sim \widetilde{O}_i$, where $\widetilde{O}_i = [\mathbf{O}_2, (u_i, 1-u_i)]$.
- Notice that $u_1 = 1$, $u_r = 0$, and $0 \leq u_i \leq 1$ for $i = 2, \ldots, r-1$. $u_2, \ldots, u_{r-1}$ need to be assessed by the DM, and the assessments describe the risk attitude of the DM.



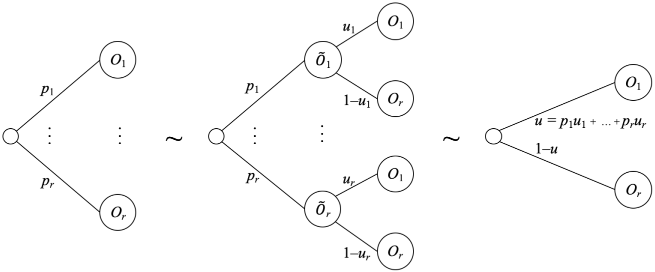
vN-M's Utility Theory

- As $O_2 \succsim O_3$, we assume that $u_2 \geq u_3$, etc. Formally, we assume:
- **Assumption** $4p$ (completeness and transitivity) The preference relation $\succsim$ for lotteries in $\mathcal{L}$ is complete and transitive.
- Assumption $4p$ generalizes Assumption $1p$ for outcomes, which can be regarded as degenerate lotteries.
- **Assumption** $5p$ (substitutability) In any lottery $L = [\mathbf{O}, \mathbf{p}]$, if we substitute an outcome O_i by the reference lottery $\widetilde{O}_i = [\mathbf{O}_2, (u_i, 1 - u_i)]$ that is indifferent to O_i, then the result is a compound lottery that is indifferent to L.



vN-M's Utility Theory

- From Assumptions $1p - 5p$, given any lottery $L = [\mathbf{O}, \mathbf{p}]$, it is possible to find a reference lottery that is indifferent to L:
  1. First we replace each O_i in L by $\widetilde{O}_i$, $i = 1, \ldots, r$.
  2. Assumption $3p$ (continuity) states these indifferent lotteries exist. Assumption $5p$ (substitutability) says they are substitutable (without changing the preference relation. By using Assumption $4p$ serially $[\mathbf{O}, \mathbf{p}] \sim [\widetilde{\mathbf{O}}, \mathbf{p}]$.
  3. By applying Assumption $2p$ (reduction of compound lottery), $[\widetilde{\mathbf{O}}, \mathbf{p}] \sim [\mathbf{O}_2, (u, 1-u)]$ where $u = \sum_{i=1}^{r} p_i\, u_i$.



vN-M's Utility Theory

- **Assumption** $6p$ (monotonicity) Suppose $L = [\mathbf{O}_2, (p, 1-p)]$ and $L' = [\mathbf{O}_2, (p', 1-p')]$. Then $L \succsim L'$ if and only if $p \geq p'$.
- Assumption $6p$ allows us to define $u(L)$ as the utility of O_1 in an indifferent reference lottery. And as argued in the previous slide, we can always find a reference lottery that is indifferent to L.

vN-M's Utility Theory

Theorem (vN-M representation theorem)

If the preference relation $\succsim$ on $\mathcal{L}$ satisfies Assumptions $1p - 6p$, then there are numbers u_i associated with outcomes O_i for $i = 1, \ldots, r$, such that for any two lotteries $L = [\mathbf{O}, \mathbf{p}]$, and $L' = [\mathbf{O}, \mathbf{p}']$, $L \succsim L'$ if and only if

$$\sum_{i=1}^{r} p_i \, u_i \geq \sum_{i=1}^{r} p_i' \, u_i$$

Thus, for $L = [\mathbf{O}, \mathbf{p}]$, we can define $u(L) = \sum_{i=1}^{r} p_i \, u_i$, where $u_i = u(O_i)$. Also, such a linear utility function is unique up to a positive affine transformation, i.e., if $u_i' = a \, u_i + b$, where $a > 0$ and b are real constants, then $u(L) = \sum_{i=1}^{r} p_i \, u_i'$ is also qualifies as a utility function.

vN-M's Utility Theory

Example 1 (Ellsberg's urn)

- Suppose we have an urn with 90 balls, of which 30 are red, and the remaining 60 are either black or yellow.
- We draw a ball at random from the urn. Let X denote the color of the ball drawn. $\Omega_X = \{r, b, y\}$. The uncertainty of X can be described by PMF P for X such that $P(r) = 1/3$, and $P(b) + P(y) = 2/3$.
- You are offered a choice between L_1 : $\$100$ on red, and L_2 : $\$100$ on black. Which one would you choose?
- L_1 can be represented by PMF P_1 for **O** $= \{\$100, \$0\}$ such that $P_1(\$100) = 1/3$, and $P_1(\{\$0\}) = 2/3$. Thus, $u(L_1) = (1/3)\, u(\$100) + (2/3)\, u(\$0)$.
- L_2 can be represented by PMF P_2 for **O** $= \{\$100, \$0\}$ such that $P_2(\$100) = P(b)$, and $P_2(\$0) = 1/3 + P(y)$. Thus, $u(L_2) = P(b)\, u(\$100) + (1/3 + P(y))\, u(\$0)$.
- Most respondents preferred L_1 to L_2. This implies that $P(b) < 1/3$.

A New Utility Theory for D-S Belief Functions

Example 1 (Ellsberg's urn, continued)

- Next, You are offered a choice between $L_3 : \$100$ on red or yellow, and $L_4 : \$100$ on black or yellow. Which one would you choose?
- L_3 can be represented by PMF P_3 for **O** $= \{\$100, \$0\}$ such that $P_3(\{\$100\}) = 1/3 + P(y)$, and $P_3(\$0) = P(b)$. Thus, $u(L_3) = (1/3 + P(y))\, u(\$100) + P(b)\, u(\$0)$.
- L_4 can be represented by PMF P_4 for **O** $= \{\$100, \$0\}$ such that $P_4(\$100) = 2/3$, and $P_4(\$0) = 1/3$. Thus, $u(L_4) = (2/3)\, u(\$100) + (1/3)\, u(\$0)$.
- Most respondents preferred L_4 to L_3. Also, the respondents who preferred L_1 to L_2 preferred L_4 to L_3. This implies that $P(b) > 1/3$, a contradiction!
- vN-M utility theory is unable to represent most respondents preferences for such lotteries.

A New Utility Theory for D-S Belief Functions

- Ellsberg [1961] spawned a large literature on theories to explain the ambiguity aversion phenomenon
- Using probability theory (unprincipled):
  - Becker and Brownson [1964], J. of Political Economy
  - Einhorn and Hogarth [1986], J. of Business
- Using credal set semantics of belief functions:
  - Gilboa and Schmeidler [1989], J. Math. Economics
  - Jaffray [1989], OR Letters
  - Gajdos et al. [2008], J. Econ. Theory

Outline

A New Utility Theory for D-S Belief Functions

- A belief function (bf) lottery is a tuple $[\mathbf{O}, m]$, where m is a basic probability assignment (BPA) for **O**, i.e., $m : 2^{\mathbf{O}} \rightarrow [0, 1]$ such that $\sum_{\mathsf{a} \in 2^{\mathbf{O}}} m(\mathsf{a}) = 1$. Here, $2^{\mathbf{O}}$ is the set of non-empty subsets of **O**.
- We assume that the outcome of a bf lottery is a single outcome, and the lottery will not be repeated.
- m is a BPA that reflects the DM's beliefs about which outcome will occur in a realization of the lottery.
- Let $\mathcal{L}_{bf}$ denote the set of all bf lotteries on **O**.
- We have a DM who has preferences on $\mathcal{L}_{bf}$, and our task is the find a utility function $u : \mathcal{L}_{bf} \rightarrow [\mathbb{R}]$ (real-valued interval) that represents DM's partial preferences on $\mathcal{L}_{bf}$.

A New Utility Theory for D-S Belief Functions

Example 1 (Ellsberg's urn)

- Suppose we have an urn with 90 balls, of which 30 are red, and the remaining 60 are either black or yellow.
- We draw a ball at random from the urn. Let X denote the color of the ball drawn. $\Omega_X = \{r, b, y\}$. The uncertainty of X can be described by BPA m for X such that $m(\{r\}) = 1/3$, and $m(\{b, y\}) = 2/3$.
- You are offered a choice between L_1 : $\$100$ on red, and L_2 : $\$100$ on black. Which one would you choose?
- L_1 can be represented by BPA m_1 for **O** $= \{\$100, \$0\}$ such that $m_1(\{\$100\}) = 1/3$, and $m_1(\{\$0\}) = 2/3$.
- L_2 can be represented by BPA m_2 for **O** $= \{\$100, \$0\}$ such that $m_2(\{\$0\}) = 1/3$, and $m_2(\{\$100, \$0\}) = 2/3$.
- L_1 and L_2 are bf lotteries.

A New Utility Theory for D-S Belief Functions

Example 1 (Ellsberg's urn, continued)

- Next, You are offered a choice between $L_3 : \$100$ on red or yellow, and $L_4 : \$100$ on black or yellow. Which one would you choose?
- L_3 can be represented by BPA m_3 for **O** $= \{\$100, \$0\}$ such that $m_3(\{\$100\}) = 1/3$, and $m_3(\{\$100, \$0\}) = 2/3$.
- L_4 can be represented by BPA m_1 for **O** $= \{\$100, \$0\}$ such that $m_4(\{\$0\}) = 1/3$, and $m_4(\{\$100\}) = 2/3$.
- L_3 and L_4 are bf lotteries.

A New Utility Theory for D-S Belief Functions

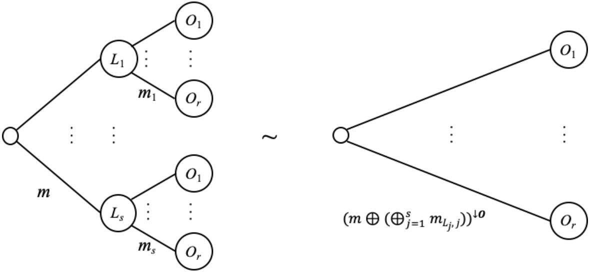
- **Assumption** $1b$ (completeness and transitivity) The preference relation $\succsim$ for **O** is complete and transitive.
- As in the probabilistic case, we assume that the outcomes are labelled such that

$$O_1 \succsim \ldots \succsim O_r, \quad \text{and} \quad O_1 \succ O_r$$

.

A New Utility Theory for D-S Belief Functions

- **Assumption** $2b$ (reduction of compound lotteries) Suppose $[\mathbf{L}, m]$ is a compound lottery, where $\mathbf{L} = (L_1, \ldots, L_s)$, m is a BPA for $\mathbf{L}$, $L_j = [\mathbf{O}, m_j]$ is a bf lottery on $\mathbf{O}$, and m_j is a conditional BPA for $\mathbf{O}$ given L_j, for $j = 1, \ldots, s$. Then, $[\mathbf{L}, m] \sim [\mathbf{O}, m']$, where $m' = (m \oplus (\bigoplus_{j=1}^{s} m_{L_j,j}))^{\downarrow \mathbf{O}}$, and $m_{L_j,j}$ is a BPA for $(\mathbf{L}, \mathbf{O})$ obtained from BPA m_j for $\mathbf{O}$ by conditional embedding, for $j = 1, \ldots, s$.



A New Utility Theory for D-S Belief Functions

- We define a reference bf lottery $[\mathbf{O}_2, m]$, where m is a BPA for $\mathbf{O}_2 = \{O_1, O_r\}$.
- **Assumption** $3b$ (continuity) Suppose $[\mathbf{O}, m]$ is a bf lottery derived from some BPA m'. Each focal element a of m (considered as a deterministic bf lottery) is indifferent to a bf reference lottery $[\mathbf{O}_2, m_\mathsf{a}]$ such that $m_\mathsf{a}(\{O_1\}) = u_\mathsf{a}$, $m_\mathsf{a}(\{O_r\}) = v_\mathsf{a}$, and $m_\mathsf{a}(\{O_1, O_r\}) = w_\mathsf{a}$, for some $u_\mathsf{a}, v_\mathsf{a}, w_\mathsf{a} \geq 0$, and $u_\mathsf{a} + v_\mathsf{a} + u_\mathsf{a} = 1$. Furthermore, $w_\mathsf{a} = 0$ if $\mathsf{a} = \{O_i\}$ is a singleton focal set of m.
- Assumption 3b is a generalization of Assumption 3p.

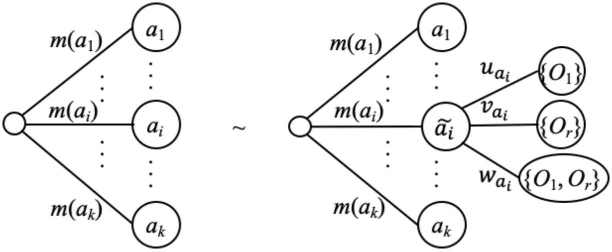
A New Utility Theory for D-S Belief Functions

Example 1 (Ellsberg's urn)

- Consider lottery $L_2 = [\{\$100, \$0\}, m_2]$, where $m_2(\{\$0\}) = 1/3$, and $m_2(\{\$100, \$0\}) = 2/3$.
- $O_1 = \$100$, and $O_r = \$0$. Focal set $\{\$0\} \sim [\mathbf{O}_2, m_{\{\$0\}}]$, where $m_{\{\$0\}}(\{\$0\}) = v_{\{\$0\}} = 1$. No assessment is required.
- To make the assessment for $\{\$100, \$0\}$, consider a DM who wishes to find a probabilistic reference lottery $[\{\$100, \$0\}, (p, 1-p)]$ that is equally preferred to $\{\$100, \$0\}$. A DM may have the following preferences: For $p < 0.2$, she prefers $\{\$100, \$0\}$, and for $p > 0.3$, she prefers the probabilistic reference lottery. She is unable to give us a precise p such that $\{\$100, \$0\} \sim [\{\$100, \$0\}, (p, 1-p)]$. For such a DM, we assess a bf reference lottery $[\{\$100, \$0\}, m_{\mathsf{a}}]$ such that $Bel_{m_{\mathsf{a}}}(\$100) = 0.2$, and $Pl_{m_{\mathsf{a}}}(\$100) = 0.3$, i.e., $u_{\{\$100,\$0\}} = 0.2$, $v_{\{\$100,\$0\}} = 0.7$, $w_{\{\$100,\$0\}} = 0.1$.

A New Utility Theory for D-S Belief Functions

- **Assumption** $4b$ (reflexive and transitive) The preference relation $\succsim$ for $\mathcal{L}_{bf}$ is reflexive and transitive.
- In comparison with Assumption $4p$, we do not assume that $\succsim$ is complete. It is neither descriptive nor normative, and consistent with D-S theory philosophy of incomplete knowledge.
- **Assumption** $5b$ (substitutability) In any bf lottery $L = [\mathbf{O}, m]$, if we substitute a focal element a_i of m by an equally preferred bf reference lottery $\widetilde{\mathsf{a}_i} = [\mathbf{O}_2, m_{\mathsf{a}_i}]$, then the result is a compound lottery that is indifferent to L.



A New Utility Theory for D-S Belief Functions

Theorem (Reducing a bf lottery to an indifferent bf reference lottery)

Under Assumptions $1b - 5b$, any bf lottery $L = [\boldsymbol{O}, m]$ with focal sets $a_1, \ldots a_k$ is indifferent to a bf reference lottery $\widetilde{L} = [\boldsymbol{O}_2, \widetilde{m}]$, such that

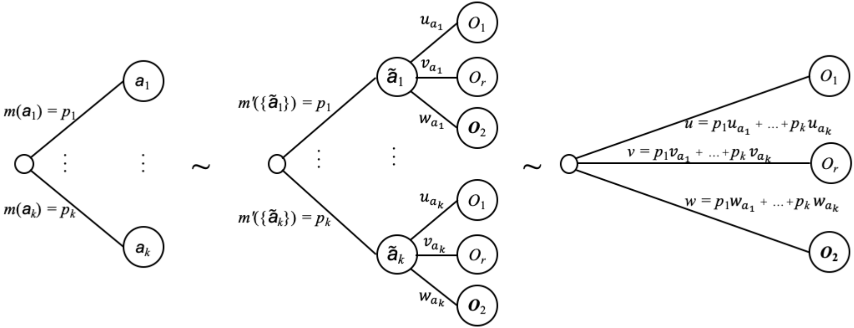
$$\widetilde{m}(\{O_1\}) = \sum_{i=1}^{k} m(a_i)\, u_{a_i}, \tag{2a}$$

$$\widetilde{m}(\{O_r\}) = \sum_{i=1}^{k} m(a_i)\, v_{a_i}, \quad \text{and} \tag{2b}$$

$$\widetilde{m}(\boldsymbol{O}_2) = \sum_{i=1}^{k} m(a_i)\, w_{a_i}, \tag{2c}$$

where u_{a_i}, v_{a_i}, and w_{a_i}, are the masses assigned, respectively, to $\{O_1\}$, $\{O_r\}$, and $\boldsymbol{O}_2$, by the bf reference lottery $\widetilde{a}_i$ equivalent to a_i.

A New Utility Theory for D-S Belief Functions



A New Utility Theory for D-S Belief Functions

- **Assumption** $6b$ (monotonicity) Suppose $L = [\mathbf{O}_2, m]$ and $L' = [\mathbf{O}_2, m']$ are bf reference lotteries, with $m(\{O_1\}) = u$, $m(\mathbf{O}) = w$, $m'(\{O_1\}) = u'$, $m'(\mathbf{O}) = w'$. Then, $L \succsim L'$ if and only if $u \geq u'$ and $u + w \geq u' + w'$.
- Thus, $L \succsim L'$ if and only if $Bel_m(O_1) \geq Bel_{m'}(O_1)$ and $Pl_m(O_1) \geq Pl_{m'}(O_1)$, i.e., if and only if outcome O_1 is both more credible and more plausible under L than L'.
- The corresponding indifference relation is: $L \sim L'$ if and only if $u = u'$ and $w = w'$.
- It is clear that $\succsim$ as defined in Assumption $6b$ is reflexive and transitive.
- Thus, L and L' are *incomparable* if one of the intervals $[u, u + w]$ and $[u', u' + w']$ is strictly included in the other.

A New Utility Theory for D-S Belief Functions

- Assumptions $1b$, $3b$, and $6b$ imply the following consistency constraints between the reference bf lotteries equivalent to singleton outcomes:

$$1 = u_{O_1} \geq u_{O_2} \geq \ldots \geq u_{O_r} = 0.$$

- Our final assumption has no counterpart in the vN-M theory.
- **Assumption** $7b$ (consistency) Let $\mathsf{a} \subseteq \mathbf{O}$, and let $\underline{O}_{\mathsf{a}}$ and $\overline{O}_{\mathsf{a}}$ denote, respectively, the worst and the best outcome in a. Then we have

$$\mathsf{a} \succsim \underline{O}_{\mathsf{a}} \quad \text{and} \quad \overline{O}_{\mathsf{a}} \succsim \mathsf{a}.$$

- Assumptions $6b$ and $7b$ imply that, for any focal sets a of m, we have

$$u_{\mathsf{a}} \geq \min_{O_i \in \mathsf{a}} u_{\{O_i\}}, \quad \text{and} \quad u_{\mathsf{a}} + w_{\mathsf{a}} \leq \max_{O_i \in \mathsf{a}} u_{\{O_i\}}. \qquad (3)$$

A New Utility Theory for D-S Belief Functions

Theorem (Interval-valued utility for bf lotteries)

Suppose $L = [\boldsymbol{O}, m]$ and $L' = [\boldsymbol{O}, m']$ are bf lotteries on $\boldsymbol{O}$. If the preference relation $\succsim$ on $\mathcal{L}_{bf}$ satisfies Assumptions $1b - 6b$, then there are intervals $[u_{a_i}, u_{a_i} + w_{a_i}]$ associated with subsets $a_i \in 2^{\boldsymbol{O}}$ such that $L \succsim L'$ iff

$$\sum_{a_i \in 2^{\boldsymbol{O}}} m(a_i)\, u_{a_i} \geq \sum_{a_i \in 2^{\boldsymbol{O}}} m'(a_i)\, u_{a_i}, \text{ and}$$

$$\sum_{a_i \in 2^{\boldsymbol{O}}} m(a_i)\, (u_{a_i} + w_{a_i}) \geq \sum_{a_i \in 2^{\boldsymbol{O}}} m'(a_i)\, (u_{a_i} + w_{a_i}).$$

Thus, for a bf lottery $L = [\boldsymbol{O}, m]$, we can define $u(L) = [u, u + w]$ as an interval-valued utility of L, with $u = \sum_{a_i \in 2^{\boldsymbol{O}}} m(a_i)\, u_{a_i}$ and $w = \sum_{a_i \in 2^{\boldsymbol{O}}} m(a_i)\, w_{a_i}$. Also, such a utility function is unique up to a strictly increasing affine transformation.

A New Utility Theory for D-S Belief Functions

- In the imprecise literature, we have lower and upper Choquet integrals defined as follows:

Definition (Choquet integrals)

Suppose we have a real-valued function $u : \mathbf{O} \to \mathbb{R}$. The lower and upper Choquet integrals of u with respect to BPA m for $\mathbf{O}$, denoted by $\underline{u}_m$ and $\overline{u}_m$, are defined as follows:

$$\underline{u}_m = \sum_{\mathsf{a} \in 2^{\mathbf{O}}} m(\mathsf{a}) \left(\min_{O_i \in \mathsf{a}} u(O_i) \right),$$

$$\overline{u}_m = \sum_{\mathsf{a} \in 2^{\mathbf{O}}} m(\mathsf{a}) \left(\max_{O_i \in \mathsf{a}} u(O_i) \right).$$

- Thus, we can regard the interval $[\underline{u}_m, \overline{u}_m]$ as an interval-valued utility of $[\mathbf{O}, m]$.

A New Utility Theory for D-S Belief Functions

- It follows from Theorem 2 and Assumption $7b$ that

$$\underline{u}_m \leq u \leq u + w \leq \overline{u}_m.$$

  where u and w are as in Theorem 3.
- Thus, the interval-valued utility defined in Theorem 3 is always included in the lower and upper Choquet integral interval-valued utility.

A New Utility Theory for D-S Belief Functions

- A special case of Theorem 3 is if we use Bayesian bf reference lotteries for the continuity assumption (Assumption $3b$), i.e., $w_{\mathsf{a}} = 0$ for all focal sets a of m.

Corollary (Real-valued utility function)

Suppose $L = [\boldsymbol{O}, m]$ and $L' = [\boldsymbol{O}, m']$ are bf lotteries on $\boldsymbol{O}$*. If the preference relation $\succsim$ on $\mathcal{L}_{bf}$ satisfies Assumptions $1b - 6b$ and if $w_{\mathsf{a}} = 0$ for all focal sets a of m and m', then there are numbers u_{a} associated with nonempty subsets a $\subseteq \boldsymbol{O}$ such that $L_1 \succsim L_2$ if and only if*

$$\sum_{\mathsf{a} \in 2^{\boldsymbol{O}}} m(\mathsf{a})\, u_{\mathsf{a}} \geq \sum_{\mathsf{a} \in 2^{\boldsymbol{O}}} m'(\mathsf{a})\, u_{\mathsf{a}}. \tag{4}$$

Thus, for a bf lottery $L = [\boldsymbol{O}, m]$, we can define $u(L) = \sum_{\mathsf{a} \in 2^{\boldsymbol{O}}} m(\mathsf{a})\, u_{\mathsf{a}}$ as the utility of L. Also, such a utility function is unique up to a strictly increasing affine transformation.

Outline

Ellsberg's Urn

Consider the bf lotteries L_1 and L_2:

Lottery	$m_i(\{\$100\})$	$m_i(\{\$0\})$	$m_i(\{\$100, \$0\})$
L_1 (\$100 on r)	$1/3$	$2/3$	
L_2 (\$100 on b)		$1/3$	$2/3$

- Given a vacuous bf lottery $[\{\$100, \$0\}, m(\{\$100, \$0\}) = 1]$, what is an indifferent bf reference lottery? For an ambiguity-averse DM, $[\{\$100, \$0\}, m(\{\$100\}) = 1/2, m(\{\$0\}) = 1/2]$ is always preferred to $[\{\$100, \$0\}, m(\{\$100, \$0\}) = 1]$. Therefore, $u_{\{\$100,\$0\}} + w_{\{\$100,\$0\}} < 1/2$.
- L_1 is a bf reference lottery with singleton focal sets. Therefore, $u(L_1) = 1/3$.
- $u(L_2) = (2/3)[u_{\{\$100,\$0\}}, u_{\{\$100,\$0\}} + w_{\{\$100,\$0\}}]$
- Thus, an ambiguity-averse DM will prefer L_1.

Ellsberg's Urn

Consider the bf lotteries L_3 and L_4:

Lottery	$m_i(\{\$100\})$	$m_i(\{\$0\})$	$m_i(\{\$100, \$0\})$
L_3 (\$100 on r or y)	$1/3$		$2/3$
L_4 (\$100 on b or y)	$2/3$	$1/3$	

- $u(L_3) = (1/3)(1) + (2/3)[u_{\{\$100,\$0\}}, u_{\{\$100,\$0\}} + w_{\{\$100,\$0\}}]$
- L_4 is a bf reference lottery with singleton focal sets. Thus, $u(L_4) = 2/3$
- An ambiguity-averse DM will prefer L_4 as

$$\frac{1}{3} + \frac{2}{3}[u_{\{\$100,\$0\}}, u_{\{\$100,\$0\}} + w_{\{\$100,\$0\}}] < \frac{2}{3}$$

  as long as $u_{\{\$100,\$0\}} + w_{\{\$100,\$0\}} < 1/2$.
- Both choices are consistent with Ellsberg's empirical findings.

One Red Ball

One red ball (Jiroušek and Shenoy 2017)

- An urn possibly contains balls of 6 colors: red (r), blue (b), green (g), orange (o), white (w), and yellow (y).
- There are n balls in the urn (n is a positive integer, and is known), and exactly one is r.
- First you pick a color, and then draw one ball at random from the urn.
- You win $\$100$ if the color of the ball drawn matches your pick, $\$0$ otherwise
- Which color do you pick?
- In informal experiments (conducted by Radim Jiroušek, subjects were graduate students familiar with vN-M's and Savage's expected utility theory), all picked r for $n = 1, \ldots, 7$. For $n \geq 8$, many chose a color different from r. Some chose r even for n as high as 11.

One Red Ball

- Let X denote the color of the ball drawn at random. The BPA m for X is as follows: $m(\{r\}) = 1/n$, $m(\{b, g, o, w, y\}) = (n-1)/n$.
- Let $L_r = [\{\$100, \$0\}, m_r]$ denote the bf lottery representing choice of color r. Then $m_r(\{\$100\}) = 1/n$, $m_r(\{\$0\}) = (n-1)/n$.
- Let $L_b = [\{\$100, \$0\}, m_b]$ denote the bf lottery representing choice of color b. Then $m_b(\{\$0\}) = 1/n$, $m_b(\{\$100, \$0\}) = (n-1)/n$.
- $u(L_r) = \frac{1}{n}$, and $u(L_b) = \frac{n-1}{n}[u_{\{\$100,\$0\}}, u_{\{\$100,\$0\}} + w_{\{\$100,\$0\}}]$.

One Red Ball

- So, L_b is strictly preferred to L_r whenever

$$\frac{n-1}{n}\, u_{\{\$100,\$0\}} > \frac{1}{n},$$

and L_r is strictly preferred to L_b whenever

$$\frac{n-1}{n}[u_{\{\$100,\$0\}}, u_{\{\$100,\$0\}} + w_{\{\$100,\$0\}}] < \frac{1}{n}.$$

Therefore, L_b is increasingly preferred to L_r with increasing n, which is consistent with empirical findings.

One Red Ball

- In our model, L_r and L_b are incomparable when

$$\frac{n-1}{n} u_{\{\$100,\$0\}} < 1/n < \frac{n-1}{n} (u_{\{\$100,\$0\}} + w_{\{\$100,\$0\}}),$$

i.e., when

$$u_{\{\$100,\$0\}} < \frac{1}{n-1} < u_{\{\$100,\$0\}} + w_{\{\$100,\$0\}}.$$

If forced to choose, a DM may just choose arbitrarily as the experiment in Jiroušek and Shenoy 2017 did not allow the respondents to express inability to choose between the two choices. Thus, the empirical findings does not provide any evidence for or against our model.

Two Urns with 1000 balls

Two urns with 1000 balls (Ellsberg, discussed in Becker and Brownson, 1964)

- There are 2 urns, each with 1000 balls numbered $1, \ldots, 1000$.
- Urn 1 has exactly one ball for each number.
- Urn 2 has unknown $(0, \ldots, 1000)$ number of balls of each number.
- One ball is to be chosen at random from an urn of your choice. If the number on the drawn ball matches a specific number, say 687, then you win $\$100$, otherwise you win nothing, i.e., $\$0$.
- Which urn do you choose?
- It is reported in [Becker and Brownson, 1964] that many respondents chose Urn 2. Why? Urn 1 has only 1 ball numbered 687, and therefore, probability of winning $\$100$ is small (0.001). Urn 2 could possibly have 0 to 1000 balls numbered 687. Therefore, choice of Urn 2 is appealing.

Two Urns with 1000 balls

- Let X_1 denote the number on the ball drawn from Urn 1, and let X_2 denote the number on the ball drawn from Urn 2.
  $\Omega_{X_1} = \Omega_{X_2} = \{1, \ldots, 1000\}$.
- m_{X_1} is a BPA for X_1 as follow:
  $m_{X_1}(\{1\}) = \ldots m_{X_1}(\{1,000\}) = 0.001$. m_{X_2} is a vacuous BPA for X_2.
- L_1 corresponding to choice of Urn 1 is $[\{\$100, \$0\}, m_1]$, where $m_1(\{\$100\}) = 0.001$, $m_1(\{\$0\}) = 0.999$.
- L_2 corresponding to choice of Urn 2 is $[\{\$100, \$0\}, m_2]$, where $m_2(\{\$100, \$0\}) = 1$.
- $u(L_1) = 0.001$, and $u(L_2) = [u_{\{\$100,\$0\}}, u_{\{\$100,\$0\}} + w_{\{\$100,\$0\}}]$.
- Consequently, $L_2 \succ L_1$ whenever $u_{\{\$100,\$0\}} \geq 0.001$, a condition that is easily satisfied. This may explain why many DMs prefer L_2 to L_1.

Outline

Summary & Conclusions

- We have proposed an axiomatic utility theory for D-S lotteries similar to vN-M's utility theory for probabilistic lotteries,
- The main difference is singleton outcomes are replaced by focal elements of m, probabilistic combination is replaced by Dempster's combination rule, and probabilistic marginalization is replaced by belief function marginalization.
- Our axiomatic theory is able to explain ambiguity attitude of human DMs that vN-M's utility theory cannot.
- While there are several probabilistic decision theories that explain ambiguity-attitude of human DMs (Becker and Brownson 1964, Einhorn and Hogarth 1986, etc.), they are not justified by simple axioms similar to vN-M's or Savage's.
- While there are many axiomatic theories using the credal set semantics of belief functions (incompatible with Dempster's rule), our axiomatic theory is the only one for D-S theory that can explain ambiguity attitude of human decision makers.